Tiling with Dominos and Bent Triominos

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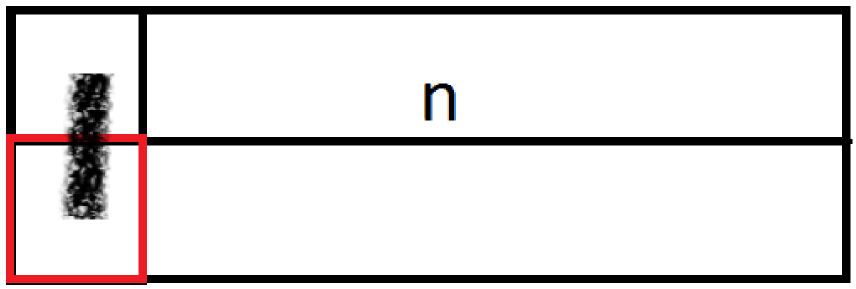
We will start by defining what it means for a board to be tileable, then we will look at what boards are tileable with both dominos and bent triominos. We are looking at gridded boards, not necessarily square, where a grid is made of squares. For a board to be tileable with either dominos or triominos, there exists a way to place the pieces such that no square in the grid is left without a piece on it.

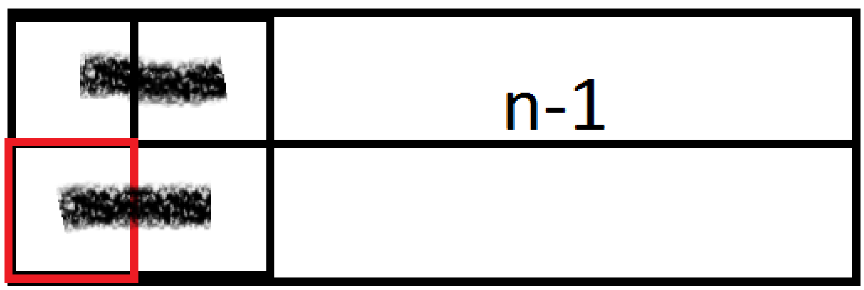
First, we are going to look at how many different ways a board of size 2 x n can be tiled with dominos. We will use the Principle of Mathematical Induction to prove that a board of size 2 x n can be tiled f_n ways, where f_j is the jth term in the Fibonacci sequence. That is, $f_j = f_{j-1} + f_{j-2}$, where $f_1 = 1$ and $f_2 = 2$.

Base step: Since we have two initial conditions, we will look at the first two numbers in this sequence. We are looking at the 2 x 1 board and the 2 x 2 board. Clearly, the 2 x 1 board can only be tiled one way and with one domino. With the 2 x 2 board, we do not have as trivial of an answer. It will take two dominos to tile it, but there are different ways that we can place these dominos. Let's look at the domino in the bottom-left corner of the 2 x 2 board. Since there must be a domino in this square, it can either be placed vertically or horizontally. If the first domino is placed vertically, then the other domino must be placed vertically, which gives us one way. If the first domino is placed horizontally, then the other domino must be placed horizontally, which gives us another way. Since there are no other variations, this gives us two ways to tile a 2 x 2 board with dominos.

Induction step: Let $n \in \mathbf{N}$ be ABF and suppose that every 2 x n board can be tiled with dominos f_n different ways and every 2 x $(n-1)$ board can be tiled with dominos f_{n-1} different ways, where f_n is the nth term in the Fibonacci sequence defined above.

Consider a board with the size 2 x $(n+1)$. Again, let's look at the bottom-left corner. There must be a domino covering this square. Recognize that the domino covering this square can be placed either vertically or horizontally. If it is placed vertically, then we are left with a 2 x n board adjacent to the vertical domino. By the induction hypothesis, this can be tiled f_n different ways. However, we can also place the domino in the bottom-left corner so that it lays horizontally. Then, another domino must be placed horizontally above it, and we are left with a 2 x $(n-1)$ board adjacent to the two already placed dominos. By the induction hypothesis, this can be tiled f_{n-1} different ways. In conclusion, the board of size 2 x $(n+1)$ can be tiled $f_n + f_{n-1}$ different ways. Here is a picture to demonstrate how we have two different options when we look at the bottom left square, which breaks the board down into a smaller 2 x n or 2 x $(n-1)$ board.

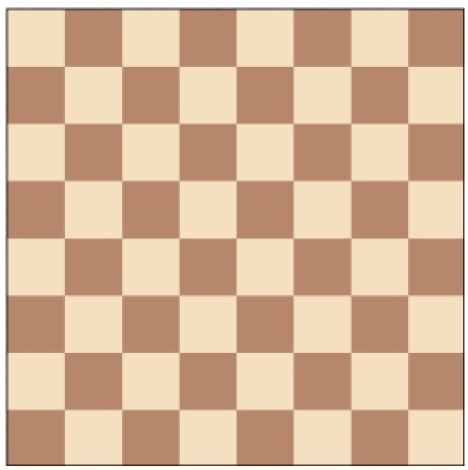




Thus, by the Principle of Mathematical induction, we can conclude that for all $n \in \mathbf{N}$, a board of size 2 x n can be tiled with dominos f_n different ways, or the nth term in the Fibonacci sequence, where the first two numbers are 1 and 2.

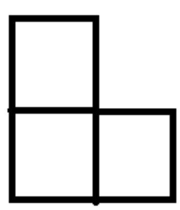
This brings us to our next topic of finding which square boards can be tiled with dominos. Immediately we can rule out boards that give us an odd number of squares. For a board to be tiled by dominos, the number of squares on the board must be divisible by two because a domino is made of two squares joined together. This is a necessary condition because it must be true, but if it is true, we still don't know if the board is tileable. Thus, when we divide the number of squares in the board by 2, we cannot have a remainder because clearly we cannot break apart the dominos to cover the remainder. Thus, since a 5 x 5 board is made up of $5^2 = 25$ squares, the 5 x 5 board cannot be tiled with dominos.

This leads us to our next board, the 8 x 8 board (chess board) with two opposite corners removed, and we want to prove that this cannot be tiled using dominos. To do this, we will use the checkerboard pattern of a regular chess board, where adjacent squares are alternating colors. All white squares are adjacent to only black squares, and all black squares are adjacent to only white squares. Since a domino is made of two squares fused together, on the regular chess board, it must cover both a black square and white square no matter where it is placed. However, if we remove opposite squares, then these squares are the same color. Here is a picture of an everyday chess board to reinforce this idea.



Notice how both pairs of opposite corners are the same color, one pair is white and one pair is brown. Thus, if we remove a pair of opposite corners, we will have 30 squares of one color, and 32 squares of the other. Since each domino must cover both colors, this board cannot be tiled using dominos.

Next, we will look at what kind of square boards with one arbitrary square removed can be tiled using *bent triominos.* Here is what a bent triomino looks like. It covers three squares in an L-shape.



As a necessary condition, since the bent triomino is comprised of three squares, we know that the m x m board with one square removed must be divisible by three, otherwise there would be no way for the triominos to fit on the board.

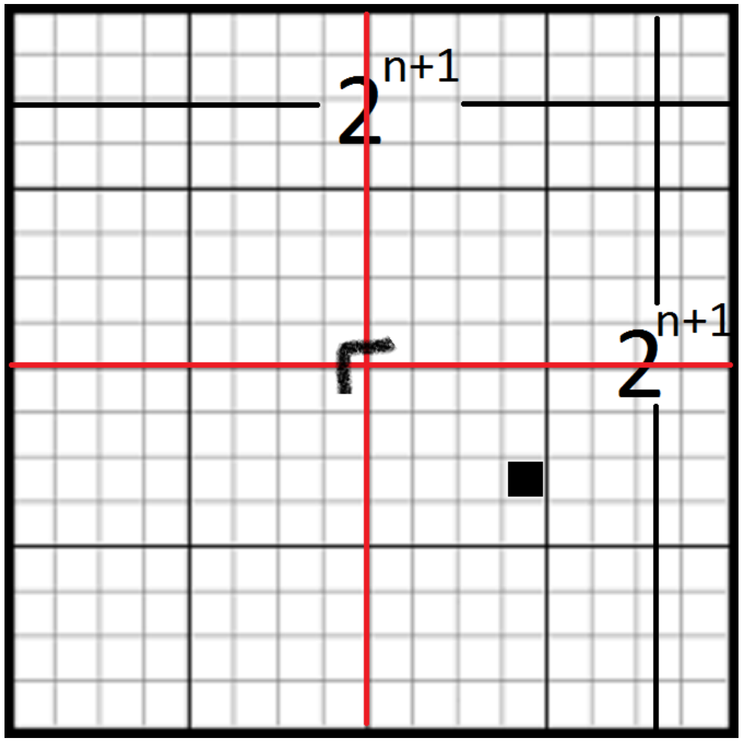
To start, we know that all 2^n x 2^n boards with one square missing are tileable with bent triominos. We can prove this using the Principle of Mathematical Induction.

Base step: For $n = 1$, we have a 2 x 2 board with one tile missing. Clearly, this can be tiled with one bent triomino.

Induction step: Let $n \in \mathbf{N}$ be ABF and suppose that every 2^n x 2^n board with an arbitrary square missing can be tiled with bent triominos.

Consider a board with the size 2^{n+1} x 2^{n+1} with an arbitrary square missing. We can split the board into four quadrants, now each with a size 2^n x 2^n, where one of the four quadrants has the missing square. Then, we can put a bent triomino in the middle of the board such that the triomino covers one square in each of the four quadrants except the one with the missing tile. Now, we have four quadrants of size 2^n x 2^n, each with one tile missing. By the induction hypothesis, each of the four 2^n x 2^n boards with one square missing can be tiled with bent triominos. Since each of the quadrants can be tiled, the entire 2^{n+1} x 2^{n+1} board with one square missing can be tiled. Thus, by the Principle of Mathematical Induction, for all $n \in \mathbf{N}$, a board of size 2^n x 2^n with one square missing can be tiled using bent triominos.

Here is a picture to demonstrate our method for tiling these boards. Note how we divide the board into 4 quadrants, locate the missing square, and place a bent triomino in the middle so that it covers the other 3 quadrants.



Now, let's look at other m x m boards with an arbitrary square removed.

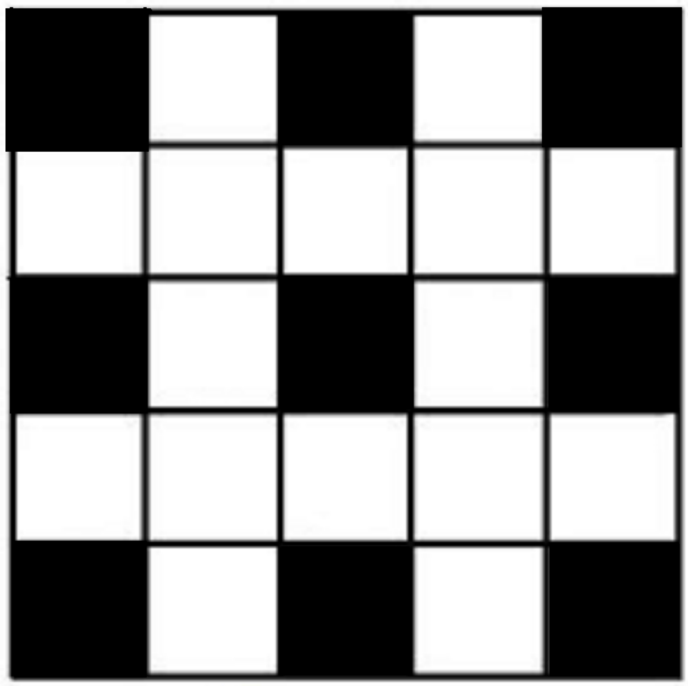
For $m = 1$ with one square removed, this is tileable using 0 tiles. In a sense, this is "vacuously tileable."

For $m = 2$ with one square removed, this is tileable using the above proof. Its dimensions are in the form 2^n x 2^n with one square removed, and in this case, $n = 1$.

For $m = 3$ with one square removed, we know this is not tileable. This is because it does not meet the necessary condition that the number of squares in its dimensions are divisible by one. There are $3^2 - 1 = 8$ squares in this board, which is not divisible by 3. Therefore, it is not tileable.

For $m = 4$ with one square removed, this is tileable using the above proof. Its dimensions are in the form 2^n x 2^n with one square removed, and in this case, $n = 2$.

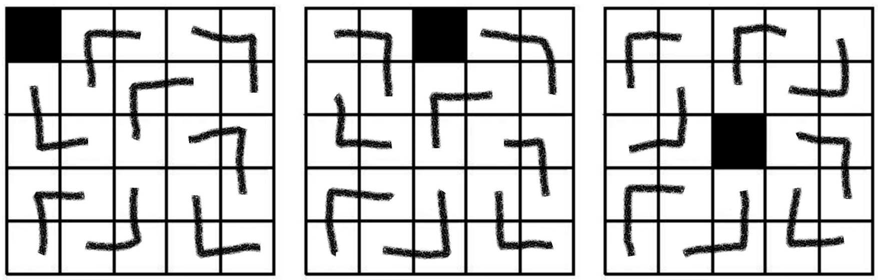
For $m = 5$ with one square removed, this is only tileable when the removed square is in the following positions:



Before we move on to other board sizes, let's consider this special board. We will now show that the 5 x 5 board can only be tiled when these special squares are removed.

First, let's look at the dimensions of the board. Since it is a 5 x 5 board with one tile missing, there are $5^2 - 1 = 24$ squares that need to be tiled. Since the triominos cover three squares each, we will use exactly 8 tiles to cover this board. Now, suppose that one of the white squares is the missing tile. In that case, the 8 triominos would have to cover the 9 black squares. Clearly, one triomino cannot cover two or more of the black squares because the black squares are not adjacent or diagonal to each other. This is guaranteed by the pigeonhole principle: there cannot be more squares that need to be tiled than triominos if one tile must be assigned to one square, such as in this case. Therefore, if any of the white squares are missing, then the 5 x 5 board cannot be tiled.

To show that if one of the black squares is removed, then the board can be tiled, we will look at the three different cases. The different types of black squares that could be removed in this case are any of the corner squares, the edge squares, or the center square. By symmetry, the corner squares are all under the same case, and the edge squares are all under the same case, so we will just show the three distinct cases: corner, edge, and center.



Thus, this shows that the 5 x 5 board with one square missing can only be tiled if the missing square is in the position of one of the 9 black squares shown on the previous page.

For $m = 6$ with one square removed, we know this is not tileable. This is because it does not meet the necessary condition that the number of squares in its dimensions are divisible by 3. There are $6^2 - 1 = 35$ squares in this board, which is not divisible by 3. Therefore, it is not tileable.

For the rest of the boards, we will use a more general argument to prove which ones are tileable. We will use the Principle of Mathematical Induction to argue that the only m x m boards with one square missing that are tileable are in the form $m = 7 + 3k$ where $k \in \mathbf{N} \cup \{0\}$ and $m = 8 + 3k$ where $k \in \mathbf{N} \cup \{0\}$.

First, let's look at the 7 x 7 board with one square missing. We will argue that this board can be tiled with bent triominos no matter which square is missing. Let this missing square's position be arbitrary but fixed. We first can cut out the 4 x 4 corner of the board that contains the missing square. By the above proof, we know that this portion of the board can be tiled using bent triominos because it is in the form 2^n x 2^n with one square removed, and in this case, $n = 2$.

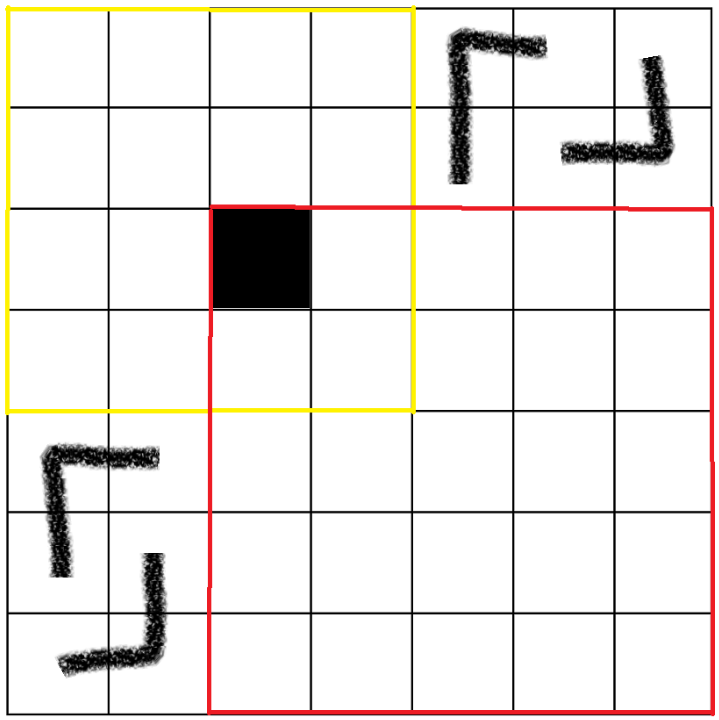
The missing square is colored in black, and the 4 x 4 grid with one square missing is outlined in yellow.

Now, we just need to tile the outer part that isn't outlined in yellow. First, let's consider the 5 x 5 board in the bottom right region. This is outlined in red. This 5 x 5 board has the upper right 2 x 2 portion missing in the sense that it is already tiled by the 4 x 4 region with one square missing. If we can prove that the 5 x 5 board with a 2 x 2 board in the corner missing is tileable, then that would help us show that the entire board is tileable.

Suppose we have a 5 x 5 board with the 2 x 2 board missing in a corner. Then, we can pretend to place a triomino in the 2 x 2 corner such that there is a missing square in the very corner of the board. Thus, we have a 5 x 5 board with one square missing, and that square is in the very corner of the 5 x 5 board. As proven before, we know that this is tileable using bent triominos. Therefore, the 5 x 5 board with a 2 x 2 corner removed is tileable using bent triominos.

So, for the entire 7 x 7 board, the portion outlined in yellow and the portion outlined in red are both tileable. To fill in the rest of the board, the 2 x 3 portions in the upper right and bottom left, we simply put two bent triominos together to create a "super triomino." We can place one of the super triominos in each of the open 2 x 3 spots, and then we have that the entire 7 x 7 board is tiled using bent triominos.

Here is the picture to demonstrate.

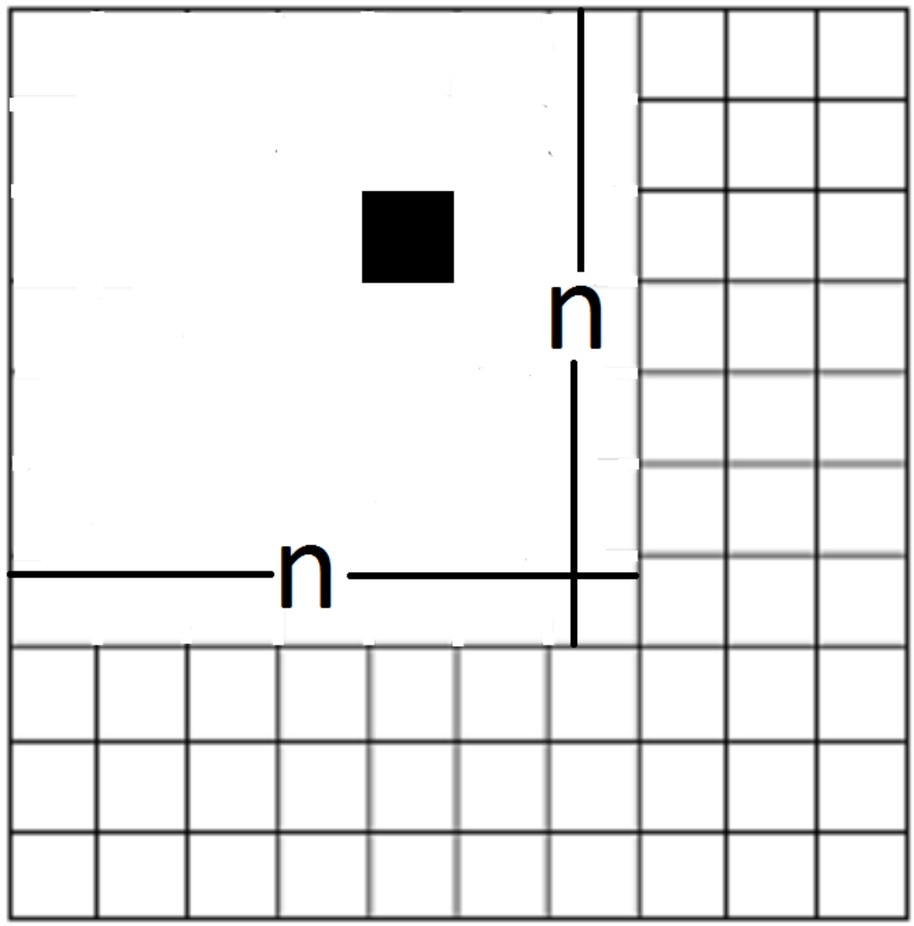


Let's consider the 8 x 8 board with one square removed as our other base case. This case is much simpler because the dimensions of the board are in the form 2^n x 2^n with one square removed, and in this case, $n = 3$. Thus, we know that the 8 x 8 board with one square removed is tileable using bent triominos.

Now that we know the 7 x 7 and the 8 x 8 boards with one square removed are tileable, we can move to our induction step. The italicized part in the statement below is our induction hypothesis, which will be crucial toward our induction argument.

Let $n \in \mathbf{N}$ be ABF, and *suppose that every n x n board with one square removed can be tiled using bent triominos.*

Now, consider a board of size $(n + 3)$ x $(n + 3)$ with one arbitrary square removed. We can then look at the square that is removed, and cut out an n x n sized board out of that region. By the induction hypothesis, we know that this can be tiled. Now, we just have to look at the strip of 3-wide tiles going along the edge, along with the 3 x 3 square in the bottom corner. Here is a picture to demonstrate what we have so far.



We have the n x n piece covering the missing square, and we know that this can be tiled. Now, if we can prove that the gridded piece surrounding the n x n grid is tileable, then the entire $(n+3)$ x $(n+3)$ board is tileable. To get started with proving that the surrounding piece is tileable, let's look at the two different cases: if n is odd or even.

In the case that n is odd, our tiling job is easy. We can cut a 4 x 4 square out of the corner opposite to the n x n square with one square of overlap that is shared by the 4 x 4 square and the n x n square. We can look at this overlapping square as a square removed from the 4 x 4 square, which is important to the tiling process. Since we have this 4 x 4 square with one square removed, we know that this can be tiled because it is in the form 2^n x 2^n with one square removed, and in this case, $n = 2$. Now that we have this tiled, we are left with 2 strips of empty squares on either side of the 4 x 4 square. These strips have dimensions $(n-1)$ x 3 and 3 x $(n-1)$. Since n is odd, $n-1$ is even, which means we can use our 2 x 3 super tiles (two bent triominos stuck together to create one 2 x 3 tile) to cover the rest of the empty squares. It will take exactly $\frac{n-1}{2}$

super tiles to cover each of the empty strips. Now, we have covered the entire board, which consists of the n x n board with the arbitrary removed square, the 4 x 4 board with one piece missing, and the two strips adjacent to the 4 x 4 piece. From this, we can conclude that for all n x n boards with one square missing that are tileable, where n is odd, we can tile all boards of size $(n+3)$ x $(n+3)$.

Now, we will look at the case where n is even. This is a bit more difficult, but not by much. Cutting a 4 x 4 piece out of the opposite corner will not work in this case because then our super tiles will have to cover an odd number of squares in their width direction. Since they have dimensions 2 x 3, they cannot cover an odd number of squares. So instead, we will cut a 5 x 5 square out of the opposite corner that is sharing 4 squares with the n x n square that is cut out, namely a 2 x 2 square. Recall from the proof of the 7 x 7 board being tileable that we can tile a 5 x 5 board that has a 2 x 2 square cut out from a corner. Again we are left with the final two strips adjacent to the 5 x 5 square that are still empty. This time, each of these strips have dimensions $(n-2)$ x 3 and 3 x $(n-2)$ since we are cutting out a 5 x 5 square instead of a 4 x 4 square. Since n is even, $(n-2)$ is also even, so these strips can be tiled with our super tiles. This time, it will take exactly $\frac{n-2}{2}$ super tiles to cover each of these remaining strips. From this, we can conclude that for all n x n boards with one square missing that are tileable, where n is even, we can tile all boards of size $(n+3)$ x $(n+3)$.

However, since we are jumping from n to $(n+3)$, we cannot conclude that all boards of size n x n are tileable for all $n \in \mathbf{N}$. Since our only base cases were for $n = 7$ and $n = 8$, we can only conclude that n x n boards with one square removed are tileable where $n = 7+3k$ where $k \in \mathbf{N} \cup \{0\}$ and $m = 8+3k$ where $k \in \mathbf{N} \cup \{0\}$.

All in all, we have concluded that the only n x n boards with one square removed that are not tileable are when n is a multiple of 3, with the exception of when $n = 5$, which is a special case in that it can only be tiled when the missing square is in one of the specified locations. That is, the n x n board with one square removed is only tileable when $n = 1, 2, 4, 5, 7, 8, \cdots$.